**Module 2: Basic Concepts of Trigonometric Functions**

**I. Right-Triangle Trigonometry**

After completing this section, you should be able to:

* find the six trigonometric function values of any acute angle of a right triangle
* determine the exact trigonometric function values of 30°, 45°, and 60° angles
* use a calculator to find trigonometric function values
* solve any right triangle by calculating its side and angle measurements
* use trigonometry to solve applied problems related to right triangles

**A. Trigonometric Ratios**

The study of trigonometry begins by focusing upon the relationships between the sides and angles of right triangles.

First consider a brief review of terminology.  
A **triangle** is a three-sided polygon.  
Each pair of sides of a triangle has a common point called a *vertex*.  
Each pair of sides forms an angle interior to the triangle.

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| Angles are often denoted by Greek letters such as sigma (alpha),  *β* (beta), *γ* (gamma), *θ* (theta), or phi (phi). | 3-sided polygon |
| Angles are often measured in degrees.  If a disk is cut into 4 equally sized wedges, each wedge contains an angle of 90 degrees. | quarter-circle quarter wedge |

If a disk were cut into 360 equal wedges, each wedge would contain an angle of 1 degree. This idea of using 360 divisions originated with the ancient Babylonians, and it remains embedded in the current degree system of angle measurement.

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| A **right angle** is an angle whose measure is 90 degrees, denoted 90°, using the "°" symbol for "degrees." | 90-degree angle |
| An **acute angle** is an angle whose measure is greater than 0° and less than 90°. | angle |
| Two angles whose measures sum to 90° are called **complementary angles**. For example, the acute angles measuring 32° and 58° are complementary, since 32° + 58° = 90°. | complementary angles |

Recall that for any triangle, the sum of the measures of the three angles is 180°.

A **right triangle** is a triangle that contains a right angle. In a right triangle, since the sum of the three angles is 180° and one of the angles is 90°, the sum of the measures of the two remaining angles must be 180° – 90°, or 90°. Therefore, the two remaining angles in a right triangle are both acute and complementary.

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| Draw a right triangle and label the acute angles  and *θ*.  The **hypotenuse** of the right triangle is the longest side, the side opposite the right angle.  The other two sides are called **legs** of the triangle. |  |
| The legs can be labeled according to their relationship to the angle *θ*.  One of the legs is the side opposite *θ*, and the other leg is referred to as the side adjacent to *θ*. |  |

The ratios of the lengths of the sides are called *trigonometric ratios*, or *trigonometric function values*, of the angle *θ*. There are six ratios.

One of the ratios is the length of side opposite *θ* divided by the length of the hypotenuse. This ratio is called the *sine* of *θ*.

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| For the right triangle shown, sin *θ* = 3/5 = 0.60 and  sin phi = 4/5 = 0.80.  This triangle is called the 3-4-5 triangle, since the lengths of the sides are 3, 4, and 5. |  |

The other five ratios are called the *cosine, tangent, cosecant, secant,* and *cotangent*. The ratios are defined as follows:

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| **Trigonometric Function Values of an Acute Angle *θ*** |
| |  |  | | --- | --- | | sin theta | cosecant theta | | cosine theta | secant theta | | tangent theta | cotangent | |

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| For the right triangle pictured, the set of six trigonometric  ratios associated with angle *θ* are listed below. |  |

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| The set of six trigonometric ratios associated with angle  are given by | |
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**B. Relationships between Trigonometric Functions**

From the definitions and previous examples, it is easy to see that there are relationships among the trigonometric function values.

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| The cosecant is the reciprocal of the sine. |  |
| The secant is the reciprocal of the cosine. |  |
| The cotangent is the reciprocal of the tangent. |  |

If you know the sine, cosine, and tangent of an angle, then by taking the reciprocals, you get the cosecant, secant, and cotangent.

There is a mnemonic aid that can help you to remember the definitions for the sine, cosine, and tangent ratios. Think of Professor SOH-CAH-TOA (pronounced phonetically as "soak a toe-a"). The joke is that the Professor is a wizard at trigonometry, but has a sore toe which needs to be soaked!

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| SOH-CAH-TOA is shorthand for | Sine: Opposite/Hypotenuse Cosine: Adjacent/Hypotenuse Tangent: Opposite/Adjacent |

For the example involving the 3-4-5 triangle, notice that the trigonometric ratios of angle *θ* are related to the trigonometric ratios of angle . For example, sin *θ* = 3/5 = cos  and cos *θ* = 4/5 = sin . Since angles *θ* and  are complementary,  = 90° – *θ*.

sin *θ* = cos  = cos (90° – *θ*) and cos *θ* = sin  = sin (90° – *θ*).

Because of this relationship, the sine and cosine functions are called *cofunctions*. There are similar relationships between the tangent and cotangent, and between the secant and cosecant functions. These relationships are listed below.

**Cofunction Identities**

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| sin *θ* = cos (90° – *θ*) | cos *θ* = sin (90° – *θ*) |
| tan *θ*= cot (90° – *θ*) | cot *θ*= tan (90° – *θ*) |
| sec *θ*= csc (90° – *θ*) | csc *θ* = sec (90° – *θ*) |

An interesting question to ask is this: For right triangles having the same proportions, does a larger triangle have larger trigonometric function values? To answer this, consider similar triangles.

Recall that two triangles are similar if the corresponding angles have the same measure or—equivalently—the corresponding sides are proportional. For example, the 3-4-5 triangle and the 6-8-10 triangle are similar. Each side in the 6-8-10 triangle is twice as long as the corresponding side in the 3-4-5 triangle.

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| **3-4-5 Triangle** | **6-8-10 Triangle** |

For the 6-8-10 right triangle, the set of six trigonometric ratios associated with angle *θ* are given by

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The 6-8-10 triangle has the same trigonometric ratios as the 3-4-5 triangle.

An analogous result applies to any two similar triangles.

Given any two similar triangles, because the lengths of corresponding sides are proportional, the trigonometric ratios remain the same. The trigonometric ratios of *θ* depend on the measure of the acute angle *θ*, but not on the size of the triangle.

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| Recall that the Pythagorean theorem (module 1, topic I-B) relates the lengths of the three sides of a right triangle. The sum of the squares of the lengths of the legs is equal to the square of the length of the hypotenuse. | https://umuc.equella.ecollege.com/file/51ed41e5-be80-4110-8171-a40ed58c98af/1/MATH108-0609.zip/Modules/M2-Module_2/images/Mod2GraphicsFiles/Mod2-SecI/B-3-Pythagorean.gif |

Given just one of the trigonometric ratios, it is possible to use the Pythagorean theorem to assist in determining the remaining five trigonometric ratios.

**Example I.B.1:** If cos sigma = 4/9 and  is an acute angle of a right triangle, what are the remaining five trigonometric function values?

**Solution:**

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| Since https://umuc.equella.ecollege.com/file/51ed41e5-be80-4110-8171-a40ed58c98af/1/MATH108-0609.zip/Modules/M2-Module_2/images/acute-tri.gif  draw a right triangle whose hypotenuse has length 9.  Label one of the acute angles , and note that the length of the side adjacent to  is 4. |  |
| Use the Pythagorean theorem to find the length of the side opposite :     Since 42 + *b*2 = 92, then *b*2 = 81 – 16 = 65 and *b* =.  Now the lengths of all three sides are known, and the five remaining trigonometric function values can be determined:   |  |  | | --- | --- | |  |  | |  |  | |  |  | | |

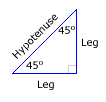
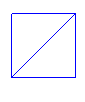
**C. Trigonometric Function Values of 30°, 45°, and 60° Angles**

There are certain acute angles that appear very frequently in trigonometric applications: 30°, 45°, and 60°. It is worthwhile to compile a table of the trigonometric function values for these angles. Fortunately, it is relatively simple to derive the values, by using basic geometric facts and working with appropriate triangles.

Recall that an **isosceles triangle** is a triangle having two sides of the same length or—equivalently—two angles of the same measure.

For an isosceles right triangle, both acute angles must have measures of 45°, since the acute angles are complementary and have equal measure.

Take a square and cut it in half along a diagonal to get two isosceles right triangles. Pick one of the isosceles right triangles and label the legs and the hypotenuse.



For convenience, suppose the legs each have length 1. Use the Pythagorean theorem to find the length of the hypotenuse:

Since 12 + 12 = *c*2, then 2 = *c*2 and the length of the hypotenuse ishttps://umuc.equella.ecollege.com/file/51ed41e5-be80-4110-8171-a40ed58c98af/1/MATH108-0609.zip/Modules/M2-Module_2/images/2sqrt.gif.

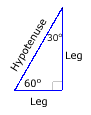
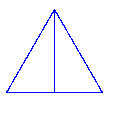
Now it is easy to determine the trigonometric function values of an angle of 45°.

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|  | The remaining trigonometric function values can be determined by taking the reciprocals. |

Next, determine the trigonometric function values of 30° and 60°. Notice that if a right triangle contains an angle of 60°, then the other acute angle must have a measure of 30°, since the angles are complementary.

An **equilateral triangle** is a triangle having all three sides of the same length, or equivalently all three angles of the same measure. Because the sum of the measures of the three angles must be 180°, each angle in an equilateral triangle must measure 60°.

Take an equilateral triangle and cut it in half to get two right triangles.



Each right triangle has acute angles 30° and 60°.

For convenience, suppose that the sides of the equilateral triangle have length 2. Then each right triangle has a hypotenuse of length 2 and a leg of length 1.

Use the Pythagorean theorem to find the length of the other leg:

12 + *b*2 = 22, so *b*2 = 3 and the length of the other leg is.

Now it is easy to determine the trigonometric function values for angles of 30° and 60°.

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|  | The remaining trigonometric function values can be determined by taking the reciprocals. |

The sine, cosine, and tangent function values of 30°, 45°, and 60° are summarized in the following table.

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| **Angle *θ*** | **sin  *θ*** | **cos *θ*** | **tan *θ*** |
| 30º | one half | squareroot of 2 over 3 | squareroot of 3 over 3 |
| 45º | squareroot of 2 over 2 | squareroot of 2 over 2 | 1 |
| 60º | squareroot of 3 over 2 | one-half | squareroot of 3 |

For the 45° angle, it is good to keep in mind the 1-1-square root of 2 triangle.

For the 30° and 60° angles, it is good to keep in mind the 1-square root of 3-2 triangle. The shortest side is the side opposite the 30° angle.

By drawing and referring to the appropriate triangle, you can quickly derive the trigonometric function values without consulting the table.

**D. Using a Calculator**

Trigonometric function values of any angle can be found using a scientific or graphing calculator. Most calculators find values for the sine, cosine, and tangent, and the user finds the cosecant, secant, and cotangent by computing the appropriate reciprocal.

As you will see later on in topic III-A, there is more than one unit of measurement for angles. Be sure to put your calculator in degree mode (denoted DEG) when working with angles measured in degrees.

**Example I.D.1:** Find sin 37.5° and csc 37.5°, rounding the results to four decimal places.

**Solution:**

sin 37.5° ≈ 0.608761429, or 0.6088 when rounded to four decimal places.

csc 37.5° = 1/(sin 37.5°) ≈ 1.642679632, or 1.6427 when rounded to four decimal places.

Recall that very large numbers and very small numbers are often written in scientific notation. Many calculators display a letter E preceding the exponent. For example, a calculator display of 1.6520000E-03 represents 1.6520000 × 10-3 = 0.001652.

**Example I.D.2:** Find tan 0.03° and cot 0.03°, writing the values in scientific notation to four decimal places.

**Solution:**

tan 0.03° ≈ 5.235988234 × 10-4, or 5.2360 × 10-4 when rounded.

cot 0.03° = 1/(tan 0.03°) ≈ 1909.859143 ≈ 1.9099 × 103 in scientific notation.

So far, you have taken an angle and found a corresponding trigonometric value. Going from a trigonometric value to an angle is the inverse process. (Inverse trigonometric functions will be discussed in detail in module 3.)

A calculator can be used to carry out the inverse process, to find an acute angle that has a given trigonometric value. Typically, a calculator uses a SHIFT key or an INV key. Put your calculator into degree (DEG) mode. Enter the trigonometric value, press SHIFT (or INV), then the trigonometric function. (You may need to consult the instructions for your calculator.)

**Example I.D.3:** Find the acute angle whose cosine is 0.8391, rounding to 2 decimal places.

**Solution:**

The acute angle whose cosine is 0.8391 is approximately 32.95°.

**E. Solving Right Triangles**

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| Consider a right triangle having legs of length *a* and *b*, a hypotenuse of length *c*, and acute angles *θ* and . |  |

To **solve a triangle** is to determine the lengths of all three sides and the measures of all three angles. Solving a triangle is similar to solving a puzzle. There are six values to find.

In the case of a right triangle, one of the angles must have a measure of 90°, and the lengths of the sides are related by the Pythagorean theorem. So, with just a little more information, it is possible to solve the right triangle.

**Problem 1:** Given a right triangle, suppose you know the lengths of any two sides. That is, suppose you know the lengths of both legs, or the length of the hypotenuse and the length of one leg. Think about how to solve the triangle. ([HINT](https://umuc.equella.ecollege.com/file/51ed41e5-be80-4110-8171-a40ed58c98af/1/MATH108-0609.zip/Modules/M2-Module_2/popups/HINT1.html))

**Example I.E.1:** Solve the right triangle having a hypotenuse of length 15 and a leg of length 8.

**Solution:**

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| Draw and label a right triangle so that the hypotenuse is 15 and one of the legs is 8.  Label the angles. Suppose *θ* is the angle whose opposite side has length 8. It is helpful to label the other acute angle and the third leg. Then at a glance, you can tell that in order to solve the triangle, you must determine the two angles and the length of the side adjacent to *θ*.  Use the Pythagorean theorem to find the length of side *a*:  *a*2 + 82 = 152, so *a*2 = 225 – 64 = 161 and  *a* = square root of 161 |  |
| Since you know the lengths of the hypotenuse and the side opposite angle *θ*, the sine of *θ* can be determined:  sin *θ* = 8/15  Using a calculator, find the acute angle *θ* whose sine is 8/15, which is approximately 32.2°.  The measure of the other acute angle https://umuc.equella.ecollege.com/file/51ed41e5-be80-4110-8171-a40ed58c98af/1/MATH108-0609.zip/Modules/M2-Module_2/images/phi-red.gif is approximately 90° – 32.2° = 57.8°.  The lengths of the three sides are 8, 15, and square root of 161. The measures of the angles are 32.2°, 57.8°, and 90°. | |

**Problem 2:** Given a right triangle, suppose you know the measure of one of the acute angles and the length of one side. That is, you know one angle and the length of either the hypotenuse or the side opposite the angle or the side adjacent to the angle.

Think about how to solve the triangle. ([HINT](https://umuc.equella.ecollege.com/file/51ed41e5-be80-4110-8171-a40ed58c98af/1/MATH108-0609.zip/Modules/M2-Module_2/popups/HINT2.html))

**Example I.E.2:** Solve the right triangle having an angle of 65.4° and opposite side of length 27.

**Solution:**

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| Draw a right triangle so that an acute angle is 65.4°, and the side opposite the angle has length 27.  Denote the length of the side adjacent to the 65.4° angle by *a*, and the length of the hypotenuse by *c*. Let https://umuc.equella.ecollege.com/file/51ed41e5-be80-4110-8171-a40ed58c98af/1/MATH108-0609.zip/Modules/M2-Module_2/images/phi-red.gif be the other acute angle. | https://umuc.equella.ecollege.com/file/51ed41e5-be80-4110-8171-a40ed58c98af/1/MATH108-0609.zip/Modules/M2-Module_2/images/Mod2GraphicsFiles/Mod2-SecI/E-9-ExE2.gif |

Since one acute angle has measure 65.4°, the other angle https://umuc.equella.ecollege.com/file/51ed41e5-be80-4110-8171-a40ed58c98af/1/MATH108-0609.zip/Modules/M2-Module_2/images/phi-red.gif has measure 90° – 65.4° = 24.6°.

Since the angle 65.4° is known and the length of the opposite side is known, the sine function can be used to find the length of the hypotenuse:

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Solve for *c*:

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The tangent function can be used to find *a*, the length of the side adjacent to 65.4°:

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Solve for *a*:

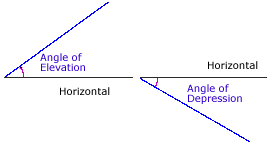
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The lengths of the three sides are 12.36, 27, and 29.70. The measures of the angles are 24.6°, 65.4°, and 90°.

**F. Applications of Right Triangles**

The branch of mathematics known as trigonometry arose from the need to solve real-world problems involving right triangles.

Applications often involve a horizontal line of reference. An angle measured above the horizontal line is called an *angle of elevation*. An angle measured below the horizontal line is called an *angle of depression*.



**Example I.F.1:** A monument and a building are 200 feet apart. At the top of the monument, the angle of elevation to the top of the building is 35°. If the monument is 80 feet high, how tall is the building?

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**Example I.F.2:** Aviation equipment aboard an airplane has determined that the distance from the plane to a radar station is 14.8 miles, and the angle of depression is 22°. If a car is parked on the ground directly below the plane's aviation equipment, how far is the car from the radar station? What is the altitude of the plane, in feet?

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